Quantum fields

Many-body problem gave birth to quantum field theory

\[ \Phi(x) \quad - \quad \text{classical field} \]
\[ \downarrow \quad \text{via second quantization} \]
\[ \hat{\Phi}(x) \quad - \quad \text{destroys a particle} \]
\[ \text{at position } x, \quad \text{fluctuates} \]

It was Einstein who in 1905 and 1907 introduced quanta of energy \( E = \frac{\hbar \omega}{2} \) for el-mag and crystal matter.

First quantization recap:

\[ p \rightarrow \hat{p} = -i \hbar \frac{\partial}{\partial x} \implies [\hat{x}, \hat{p}] = i \hbar \]

Heisenberg uncertainty rel.

\[ \Delta x \Delta p \geq \frac{\hbar}{2} \]

We want however treat a many-body problem and usually Sch. eq is useless.

Instead, we are interested in 1) excitations above ground state, 2) correlations and response to ext. probes 3) collective behavior.
Quantization of a classical string:

\[ H = \int dx \left[ \frac{1}{2} (\nabla_x \phi)^2 + \frac{1}{2g^2} \hat{x}^2(x) \right] \]

\[ [\hat{\phi}(x), \hat{x}(y)] = i \hbar \delta(x-y) \]

**Quantum statistics recap:** In QM there are two fundamentally different types of particles - fermions and bosons.

\[ \psi(\vec{p}', \vec{p}'') = e^{i\theta} \psi(\vec{p}'', \vec{p}') \]

\[ e^{i\theta} = +1 \quad \text{bosons} \]

\[ e^{2i\theta} = -1 \quad \text{fermions} \]

Double exchange \( e^{2i\theta} = +1 \)
In 1927, Jordan and Klein proposed second quantization:

\[ \psi(x) \rightarrow \hat{\psi}(x) \]

\( \hat{\psi} \) and \( i \hat{\psi}^+ \) is a pair of canonically conjugate variables

\[ H = \frac{i}{2} \hat{\psi}^+ \frac{\partial}{\partial t} \hat{\psi} \]

But how to distinguish bosons and fermions?

For bosons:

\[ [\hat{\psi}(x), \hat{\psi}^+(y)] = \delta(x-y) \]

For fermions:

\[ [\hat{\psi}(x), \hat{\psi}^+(y)] = \delta(x-y) \]

anticonnection \( \{ \hat{\psi}(x), \hat{\psi}(y) \} = 0 \)

To understand why it is so, we make connection to many-body wave-function.

First, introduce vacuum state \( |0\rangle \):

\[ \hat{\psi}(x) |0\rangle = 0 \]

a state with no particles.

Now we can create particles with \( \hat{\psi}^+ \)

\[ |x_1, x_2, \ldots, x_n\rangle = \hat{\psi}^+(x_n) \ldots \hat{\psi}^+(x_1) |0\rangle \]

and corresponding 6\( \delta \)a

\[ \langle x_1, \ldots, x_n | = \langle 0 | \hat{\psi}(x_1) \ldots \hat{\psi}(x_n) \]
The $N$-body wavefunction for a state $|N\rangle$

$$\Psi(x_1, \ldots, x_N) = \langle 0 | \hat{\Psi}(x_1) \ldots \hat{\Psi}(x_N) | N \rangle$$

Wavefunction encodes matrix elements of quantum fields

Now we can understand why fermion operators anti-commute

$$\Psi(x_1, x_2) = -\Psi(x_2, x_1)$$

$$\Psi(x_1) \Psi(x_2) = -\Psi(x_2) \Psi(x_1)$$

So, quantum statistics dictates (anti)commutation relations of fields

Out of elementary field operators we can construct more complicated operators, e.g. density operator

$$\hat{\rho}(x) = \hat{\Psi}^+(x) \hat{\Psi}(x)$$

whose expectation value in a state $|N\rangle$

$$\rho(x) = \langle N | \hat{\rho}(x) | N \rangle$$
Unlike classical fields, quantum fields fluctuate and so the canonical pair $\hat{\phi}, \hat{\phi}^\dagger$ cannot have a sharp expectation value. Let’s introduce 

$$\hat{\varphi}(x) = \sqrt{\rho(x)} e^{i\hat{\Theta}(x)}$$

for bosons e.m. relations of $\varphi, \varphi^\dagger$

$$[\hat{\rho}(x), \hat{\Theta}(y)] = i\delta(x-y)$$

which implies $\Delta N \Delta \Theta \geq 1$, we will see later that in a superfluid $\Delta N$ is large $\Rightarrow \Delta \Theta \to 0$, i.e. the phase field $\hat{\Theta}$ acquires macroscopic coherence. Similar to laser in superfluids we can observe interference!

Now we will demonstrate that field equations of quantum fields imply many-body Schrödinger equation.
\[
\hat{H} = \sum_x \hat{\psi}^+ \left(-\frac{\hbar^2 \nabla^2}{2m} + U(x)\right) \hat{\psi} + \frac{1}{2} \int \int V(x-x') \hat{\rho}(x) \hat{\rho}(y) \ dx \ dx'\ 
\text{Int. potential}
\]

where in normally ordered operators: all destruction operators are on the right:
\[
\hat{\rho}(x) \hat{\rho}(y) = + \left( \psi^+(x) \psi^+(y) \right) \psi(x) \psi(y)
\]

depends on statistics

Using Heisenberg equation of motion

\[
i \hbar \partial_t \hat{\psi} = [\hat{\psi}, \hat{H}]
\]

\[
i \hbar \partial_t \hat{\psi}(x) = \left[-\frac{\hbar^2 \nabla^2}{2m} + U(x)\right] \hat{\psi}(x) + \int dx' V(x-x') \hat{\rho}(x') \hat{\psi}(x)
\]

Now let's take \(i\hbar \partial_t\) of many-body wavefunction

\[
i \hbar \partial_t \Psi(x_1, \ldots, x_N) = i \hbar \sum_{j=1}^N \left( \psi^+(x_j) \partial_t \psi(x_j) + \psi^+(x_j) \psi(x_j) \partial_t \right) \Psi(x_1, \ldots, x_N)
\]

\[
= \sum_{j=1}^N \left[-\frac{\hbar^2 \partial^2}{2m} + U(x_j)\right] \Psi + \sum_{j=1}^N \int dx' V(x-x_j) \psi^+(x_j) \psi(x_j) \Psi(x_1, \ldots, x_N)
\]
Now we come to the density to the left
\[
\langle 0 | \hat{\psi}(x_1) \cdots \hat{\psi}(x_j) \hat{\psi}(x_{j+1}) \cdots \hat{\psi}(x_N) | \Omega \rangle = \sum_{e<j} \delta(x_i - x_e) \langle 0 | \hat{\psi}(x_1) \cdots \hat{\psi}(x_{j-1}) | e \rangle \langle e | \hat{\psi}(x_{j+1}) \cdots \hat{\psi}(x_N) | \Omega \rangle
\]

As a result, we get many-body Schrödinger equation
\[
\hat{H} \Psi(t) = \left( \sum_{j=1}^{N} H^{(0)}_j + \sum_{e<j} V_{ej} \right) \Psi(t)
\]
The final result does not depend on statistics

Grand canonical ensemble from 2nd quantization

Grand canonical ensemble - a system is in contact with heat bath which it can exchange energy and particles.

Probability of being in a state \( \lambda \) of energy \( E_\lambda \) and particle number \( N_\lambda \)
\[
\rho_\lambda = \frac{1}{Z} e^{-\beta (E_\lambda - \mu N_\lambda)}
\]
\[
Z = \sum_\lambda e^{-\beta (E_\lambda - \mu N_\lambda)} \quad \text{normalization constant}
\]