**Luttinger liquids**

In 1d no single-particle excitations, but necessarily collective excitations

2d or higher 1d

We will see that the Landau Fermi liquid paradigm breaks down in 1d and must be replaced by a different theory.

**Particle-hole excitations in 1d**

\[
\begin{align*}
E_{ph}(q) &= \varepsilon(k+q) - \varepsilon(k) \\
&= \frac{q^2}{2m} + \frac{kq}{m} \\
E_{max}(k) &= \varepsilon(k+q) - \varepsilon(k) = \frac{q^2}{2m} + \frac{kq}{m} \\
E_{min}(k) &= \varepsilon(k) - \varepsilon(k+q) = -\frac{q^2}{2m} + \frac{kq}{m}
\end{align*}
\]
particle-hole excitations become sharp as $q \to 0$

$$\frac{\Delta E_{ph}(q)}{E_{ph}(q)} \to 0$$

Contrary to higher dimensions, in 1d particle-hole excitations become sharp as $q \to 0$

$\text{Ph excitation is bosonic, first glimpse of bosonization.}$

Here we will mainly follow T. Giamarchi "Quantum physics in 1d."

See also C. Kane, Boulder Summer School lecture notes on bosonization, 2005

Tomonaga-Luttinger model

Consider an idealized relativistic model
Two models are the same at low energies, but differ away from the Fermi surface.

\[ H_{TL} = \sum_{k \in (k_{F}, -k_{F})} \Theta_{F} (S_{k} k - k_{F}) \, C_{k}^{+} C_{k} \]

When all negative energy states are filled, Dirac see

Particle-hole excitations in TL model:

\[ E_{ph}(q) = \Theta_{F}(k_{F} + q) - \Theta_{F} k = \Theta_{F} q \]

At low momentum the bosonic p-h excitation is sharp

\[ 2k_{F} q \] in this model, it cannot decay
The main idea of the LETINGER theory is to use bosonic language to describe exc.
density \[ \rho(x) = C^+(x) C(x) \]
for fermions
\[ \rho^+(q) = \sum_k C^+_{k-q} C_k \]
convolution
\[ \rho(-q) \]
sums of p-h excitations

Our plan: write \( \rho(q) \) in terms of \( \rho^+_q \)
\[ \rho^+(q) = \# b_q + \# b^+_q \]
and express \( H_{\text{TL}} \) in terms of \( 6, 6^+ \)

- surprisingly, the result will be quadratic.

In addition, in this language density-density interactions, that are quartic in fermions
\[ H_{\text{int}} \sim \frac{1}{V} \sum_q V(q) \rho(-q) \rho(q) \]
\[ \sim \frac{1}{V} \sum_q V(q) (\# b_q + \# b^+_q)^2 \]
are also quadratic in terms of \( 6, 6^+ \)
and thus \( H_{\text{int}} | H_{\text{TL}} + H_{\text{int}} \) can be
easily diagonalized.
Mathematically, need to treat Dirac see carefully. To avoid infinities introduce Normal ordering: $\langle 0 | : O : | 0 \rangle = \delta_{\text{Dirac see}}$

To move all creation/annihilation left/right by construction $\langle 0 | : O : | 10 \rangle = 0$

For $A, B$ a linear combination of creation and destruction operators

\[
: A B : = \hat{A} \hat{B} - \langle 0 | \hat{A} \hat{B} | 10 \rangle
\]

Consider how an example:

\[
: g_{\mu}(x) : = : C_{-\mu}(x) C_{\mu}(x) :
\]

We go now to Fourier space

\[
: g_{\mu}(x) : = \frac{1}{V} \sum_{p} : g_{\mu}(p) : e^{ip \cdot x}
\]

\[
: \tilde{g}_{\mu}^{+}(p) : = \left\{ \begin{array}{ll}
\sum_{k} C_{1}^{+}, k + p C_{\mu, k} & \text{if } p \neq 0 \\
\sum_{k} C_{\mu, k} C_{1, k} - \langle 0 | C_{1, k}^{+} C_{1, k} | 10 \rangle & \text{if } p = 0
\end{array} \right.
\]

finiti matrix elements

\text{final after sanction}
Now we want to compute the commutator
\[ [\rho^+(p), \rho^+(p')] \]
it is nontrivial only if \( p = p' \). First we do a naive calculation,

\[ [\rho^+(p), \rho^+(p')] = \sum_{k_1, k_2} \left[ \begin{array}{ccc} C^+_{k_1 + p} C_{k_1} \, & C^+_{k_2 - p} C_{k_2} \end{array} \right] \]

\[ = \sum_{k_1, k_2} \left( C^+_{k_1 + p} C_{k_1} \, \delta_{k_1, k_2 - p} \, - \, C^+_{k_2 - p} C_{k_1} \, \delta_{k_1 + p, k_2} \right) \]

\[ = \sum_{k_2} \left( C^+_{k_2 + p - p} C_{k_2} \, - \, C^+_{k_2 - p} C_{k_2 - p} \right) \]

if we replace \( k_2 \to k_2 + p \) in the second sum, it seems that the commutator vanishes. We must be careful with \( \infty \) in matrix elements:

\[ [\rho^+(p), \rho^+(p')] = \sum_{k_2} \left( C^+_{k_2 + p - p} C_{k_2} \, - \, C^+_{k_2 - p} C_{k_2 - p} \right) \]

\[ + \sum_{k_2} \langle 0 | C^+_{k_2 + p - p} C_{k_2} | 0 \rangle \, - \, \langle 0 | C^+_{k_2 - p} C_{k_2 - p} | 0 \rangle \]

the normal ordered contribution vanishes since all matrix elements are finite.
As a result we find:
\[ [\hat{p}^+_r(p), \hat{p}^+_r(-p')] = \delta_{pp'} \sum_{k,k'} \left( \langle 0 | c^+_r k_2 c^+_r k_2 | 0 \rangle - \delta_{rr'} \langle 0 | c^+_r k_2 - p c^+_r k_2 - p | 0 \rangle \right) \]

Consider now periodic BC \( k = \frac{2\pi n}{L} \)

if \( q \) is occupied \( \langle 0 | c^+_r q q c^+_r q | 0 \rangle = 1 \)

\[ [\hat{p}^+_r(p), \hat{p}^+_r(-p')] = -\delta_{rr'} \delta_{pp'} \frac{S_r p L}{2\pi} \]

this resembles boson commutation relation \( [b, b^+] = 1 \) (up to normalization)

in addition \( \hat{p}^+_r(p>0) | 0 \rangle = \hat{0} \hat{p}^+_r(p<0) | 0 \rangle = \hat{0} \)

act as annihilation operators on Dirac sea

We define now for \( p \neq 0 \)
\[ b^+_p = \sqrt{\frac{2\omega}{L |p|}} \sum_{s} \theta(s_r p) \hat{p}^+_r(p) \]
\[ b^+_p = \sqrt{\frac{2\omega}{L |p|}} \sum_{s} \theta(s_r p) \hat{p}^+_r(-p) \]

\[ [b^+_p, b^+_{p'}] = \delta_{pp'} \] canonical relation
Now write everything in terms of $G$s.

One can show

\[
\begin{align*}
\{6_p, \eta_\lambda\} &= \delta_\varepsilon p \ 6_p \\
H_{TL} &= \sum_{p \geq 0} \eta_\varepsilon |p| \ 6^+_p \ 6_p
\end{align*}
\]

Unexpected result: in bosonic language free Dirac theory is quadratic in $6$ and $6^+$! Since interactions are quadratic in $p$, they are easy to introduce, we get still quadratic theory?

? Can we write elementary fermions in terms of bosonic operators?

use

\[
\begin{align*}
\{p^+_p(p), c_r(x)\} &= \frac{1}{\sqrt{V}} \sum_{k_1, k_2} e^{i k_2 p} \left[ c^+_r \k_1 , p c_r \k_1, c^+_r \k_2 \right] \\
\eta_\varepsilon &= -e^{ip^+_p \cdot x} c_r(x) \\
c_r(x) &= e^{\sum_p e^{i p^+_p (\cdot p) \left( \frac{2 \pi \hbar}{p L} \right)} p^+_p (-p)}
\end{align*}
\]

non-trivial transformation
Subtlety: Rus of the last equation does not change fermion particle number, but \( c_0 \) should do that, One can introduce an operator \( \mathcal{O}_N \) (Klein factor) which suppresses charge uniformly in space and change total \( \Phi \# \# \) by \( 1 \). This operator is not important for calculation of space-time dependence of correlation functions, see Giamarchi book for details.

We succeeded in bosonization, but can we go beyond the idealized TL model? Instead of working with \( 6, 6^* \), it is convenient to work with two real angular fields \( \Phi \) and \( \Theta \)

\[ \phi \rightarrow \text{density fluctuaions} \]

\[ \Theta \rightarrow \text{phase fluctuations} \]
\[ \phi(x) = \left[ \rho_0 - \frac{1}{\text{const.}} \nabla \phi \right] \sum e^{i\nabla(x - \phi(x))} \]

\[ \rho_0 \text{- background} \]

\[ \rho_{\text{const.}} \text{ at } q \ll k \]

Localized particle = \( \mathcal{N} \text{ kink of } \phi \)

\[ \psi^+(x) = \sqrt{\rho(x)} e^{-i\Theta} \text{ for pure field} \]

1) We will show below that \( \Theta \) and \( \nabla \phi \) are canonically conjugate fields.

2) The facts on macroscopic scales (\( l \gg k^{-1} \)) and the encode universal long-wave length physics beyond the TL model into Luttinger liquid paradigm.

Calculation of commutator of \( \Theta \) and \( \phi^+ \):

First start from elementary bosonic particle:

\[ [\psi^+_n(x), \psi^+_m(x')] = \delta(x - x') \]

\[ \psi^+_n = \sqrt{\rho} e^{-i\Theta} \]
\( \mathbf{g} = \mathbf{P} \mathbf{a}, \mathbf{a} \mathbf{m} \), form the adjoint
\[
[p, q, \theta] = i \delta (x-x')
\]
Using now that at low moment
\[
\bar{\mathbf{g}}(\mathbf{x}) \propto \mathbf{p} - \frac{1}{\lambda_c} \mathbf{\nabla} \mathbf{\phi}
\]
we get
\[
[i \frac{1}{\lambda_c} \mathbf{\nabla} \mathbf{\phi}_y, \theta(x')] = - i \delta (x-x')
\]
for more careful derivation see Gianorbi.

We thus found:

1) Canonical moment of \( \phi(x) \) is
\[
\mathbf{\Pi}(x) = \frac{i}{\lambda_c} \mathbf{\nabla} \mathbf{\theta}
\]
2) \([ \phi(x), \theta(x') ] = - i \frac{1}{2 \lambda_c} \mathbf{\Delta} \mathbf{\phi}(x-x')\]

Elementary fields in terms of \( \phi \) and \( \theta \):
\[
\psi^+_B(x) = \sqrt{p_0 - \frac{1}{\lambda_c} \mathbf{\phi}} \sum_n e^{2i n (\pi \rho_0 x - \phi(x))} e^{-i \theta(x)}
\]
A boson we can also get the representation of fermion by attaching JW string
\[
\psi^+_w(x) = \psi^+_B(x) e^{i \int_0^x \mathbf{p}(y) \mathbf{dr}} = \psi^+_B(x) e^{i \int \mathbf{p}_B - \frac{1}{\lambda_c} \mathbf{\phi}}
\]
\[
\psi^+_B(x) = \sqrt{p_0 - \frac{1}{\lambda_c} \mathbf{\phi}} \sum_n e^{i (2n+1) (\pi \rho_0 x - \phi(x))} e^{-i \theta(x)}
\]
\( \theta \) and \( \phi \) can be related to \( b^+ \) and \( b \) derived before.

Hamiltonian of the Lettinger liquid in free theory

\[
\mathcal{H}_{\text{kin}} = \int \frac{\partial \psi^* \partial \psi}{2m} + \frac{\rho_0}{2m} \int (\partial \theta)^2
\]

- density-density \( \rho_+ \) = \( \mathcal{H}_{\text{kin}} + \int (\partial \phi)^2 \)

Mixed terms \( \sim \partial \phi \partial \theta \) are prohibited by the inversion symmetry \( x \to -x \).

Most general Lettinger liquid Hamiltonian

\[
\mathcal{H} = \frac{1}{2\pi} \int d\tau \left[ u \kappa \left( \frac{\partial \Pi(x)}{\partial x} \right)^2 + \frac{\kappa}{\kappa} \left( \frac{\partial \phi}{\partial x} \right)^2 \right]
\]

and the corresponding action

\[
S = \frac{1}{2\pi K} \int d\tau d\tau' \left( \frac{1}{\omega} \left( \partial_\tau \phi \right)^2 + u \left( \partial_\tau \phi \right)^2 \right)
\]

\( u \) - velocity

\( k \) - Lettinger parameter

\( K = \left\{ \begin{array}{ll} > 1 & \text{attractive} \\ < 1 & \text{repulsive} \end{array} \right. \)
This is the most general structure for interacting spinless (either bosonic or fermionic model) in 1d. While in the relativistic TL model \( w \) and \( K \) can be determined exactly in terms of microscopics, in other 1d models these parameters are difficult to compute (can be measured experimentally). Correlation functions and thermodynamics.

\( \phi, \theta \) - dimensionless angles
Since the action is quadratic in \( \phi \), we can extract \( \langle \phi \phi \rangle \) for it.

First, introduce \( \eta = 2n \)

\[
S = \frac{1}{2\alpha k} \int dx dy \left( (\partial_y \phi)^2 + (\partial_x \phi)^2 \right)
\]

in Fourier space \( x,y \sim \vec{q} \).