Superfluids and superconductors

Two related phenomena which however differ substantially at low energies

Superconductivity - discovered in the lab of Kamerlingh Onnes in 1911

many-body phenomena of electrically charged particles, such as electrons
* below $T_c$ - strictly zero resistivity
* response to magnetism:
  type I - expulsion - Meissner effect
  type II - Abrikosov lattice of vortices

quantum mechanics is essential
high $T_c$ SC: largest $T_c \approx 140 \text{ K}$ at 60 atmospheres

debates of SC mechanism

Superfluidity - many-body phenomena of electrically neutral particles, e.g. atoms
Experimentally discovered by Kapitsa and Allen in 1937
in fact Kameligh Dunes 196
cooled He below 2K, but they did not recognize its significance.
* fluid flow with no resistance
* under rotation - lattice of quantized vortices

1937 - 4He, 1972 - 3He,
1995 - cold atom BEC, 2000 - Fermionic superfluids

Despite similarities between SC and SF, at low energies they are very different
SF - gapless due to spontaneous symmetry breaking (SSB) of global U(1) particle number symmetry
SC - gapped due to the Higgs mechanism of local U(1) electric "symmetry"
Both however exhibit an energy gap in the spectrum of fermionic excitations. This gap is explained by the BCS theory.

**Basics of BCS theory**

We start from a simple model of fermions with short-range attractive interactions: \( g > 0 \)

\[
\hat{H} = \sum \left( -\frac{\Delta}{2\hbar} - \mu \right) \hat{\Psi}_\sigma^\dagger \hat{\Psi}_\sigma - \frac{g}{\sqrt{2}} \left( \hat{\Psi}_\uparrow^\dagger \hat{\Psi}_\downarrow^\dagger \hat{\Psi}_\downarrow \hat{\Psi}_\uparrow \right)
\]

Cooper discovered that in the presence of a rigid Fermi sphere, any attractive interaction \( g > 0 \) gives rise to a **bound state** (Cooper pair). This is a nontrivial consequence of kinematics of the Fermi surface (finite density of states). In vacuum (\( \mu = 0 \)) one needs a finite \( g > 0 \) in 3d to create a two-body bound state.
Cooper result suggests that the Fermi sea is unstable with respect to formation of Cooper pairs $\rightarrow$ BCS wave function.

The essence of the BCS approach: imagine that in the ground state

$$- \frac{g}{2} \langle \hat{\Psi}_i \hat{\Psi}_j \rangle_{GS} = \Delta \neq 0$$

Cooper pair operator has finite expectation value in the GS. Notice that the GS does not have fixed number of particles.

**General analysis:**

$\hat{\Psi}_i$ - second-quantized fermion operator destroys a fermion in a quantum state $i$

$i$ is spin, momentum, something else $i = 1, \ldots N$

$\{ \hat{\Psi}_i, \hat{\Psi}_j \} = \{ \hat{\Psi}_i^+, \hat{\Psi}_j^+ \} = 0 \quad \{ \hat{\Psi}_i, \hat{\Psi}_j^+ \} = \delta_{ij}$

Mean-field approximation: $\hat{\Psi}_i \hat{\Psi}_j = \Delta + (\hat{\Psi}_i \hat{\Psi}_j - \Delta)$

$$\hat{H}_{MF} = \hat{\Psi}_i^+ H^0 \hat{\Psi}_i + \frac{1}{2} \left( \Delta_{ij} \hat{\Psi}_i^+ \hat{\Psi}_j^+ + \Delta_{ij} \hat{\Psi}_i \hat{\Psi}_j \right) + \frac{1}{g_0} \Delta_{ij} \delta_{ij}$$

Hermitean matrix | anti-sym. matrix
Hne does not conserve the particle number.

It is quadratic and can be diagonalized using the Namba trick:

\[ H_{\text{ef}} = \frac{1}{2} \left( \langle \psi_i^+ | \psi_i \rangle \left( \begin{array}{cc} H_0 & \Delta_i^+ \\ \Delta_i & -H_0^T \end{array} \right) \right) \left( \begin{array}{c} \psi_i^+ \\ \psi_i \end{array} \right) + \frac{1}{2} \text{tr} H^2 + \frac{1}{2} \Delta^+ \Delta 

 \]

Now solve the single-particle BdG problem:

\[ \left( \begin{array}{cc} H_0 & \Delta \\ \Delta^T & -H_0^T \end{array} \right) \left( \begin{array}{c} \vec{U}_m \\ \vec{\sigma}_m \end{array} \right) = E_m \left( \begin{array}{c} \vec{U}_m \\ \vec{\sigma}_m \end{array} \right) \]

The BdG problem has particle-hole symmetry - for every \( E_m > 0 \) there is a corresponding \( E_m < 0 \) solution:

\[ \left( \begin{array}{cc} H_0 & \Delta \\ \Delta^T & -H_0^T \end{array} \right) \left( \begin{array}{c} \vec{U}_m^* \\ \vec{\sigma}_m^* \end{array} \right) = -E_m \left( \begin{array}{c} \vec{U}_m^* \\ \vec{\sigma}_m^* \end{array} \right) \]

Consider the translation - in 1D problem with \( E_n = \frac{k^2}{2m} - \frac{\Delta^2}{2m} \), \( \Delta = \Delta^0 \)

\[ E_k = \sqrt{E_n^2 + \Delta^2} \]
SC opens an energy gap in the fermionic spectrum, the Fermi surface is destroyed, cannot use the Landau Fermi liquid theory. Use now a linear Bogoliubov transform

\[ U^+ H_{\text{Bog}} U = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} \]

\[ U = \begin{pmatrix} U_{\text{im}} & \varphi_{\text{im}}^* \\ \varphi_{\text{im}} & U_{\text{im}}^* \end{pmatrix} \]

\[ \hat{H}_{\text{HF}} = \frac{1}{2} \hat{\mathcal{F}}^+ U \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix} U^{-1} \hat{\mathcal{F}} + \ldots \]

\[ B = (b_m^*, b_m) \]

\[ = \sum_{m=1}^{N} E_m \hat{b}_m^+ \hat{b}_m - \frac{1}{2} \sum_{m=1}^{N} E_m + \frac{1}{2} \epsilon \sigma^2 + \frac{1}{g} \sigma^+ \sigma \]

sums only over positive spectrum

\[ b_m^+ - \text{creation operator of fermionic quasi-particle with } E_m \gg \]

BCS vacuum: \( \hat{b}_m |\text{BCS}\rangle = 0 \) for any \( m \)

\[ |\text{BCS}\rangle = \prod_{m=1}^{N} |0\rangle, \text{ FSC vacuum} \]

by anticommutation relations \( \{ b_m, b_n \} = 0 \) this state is annihilated by all \( b_m \).
Expectation values: $\Theta = \hat{\Phi}_i^+ \Theta_{ij} \hat{\Phi}_j$

$$\langle \hat{\Theta} \rangle_{\text{BCS}} = \langle \text{BCS} | \hat{\Phi}_i^+ \Theta_{ij} \hat{\Phi}_j | \text{BCS} \rangle =$$

$$= \langle \text{BCS} | (\Phi_{im} \Phi_m + \Phi_{im}^* \Phi_m^*) \Theta_{ij} (\Phi_{jn} \Phi_n + \Phi_{jn}^* \Phi_n^*) | \text{BCS} \rangle =$$

$$\Phi_{im} \Phi_{ij} \Phi_{jn} \Phi_{jn}^*$$