Boson-vortex duality

We emphasized that superconductors and superfluids are very different at low energy (gapped vs gapless), but there is an intriguing relation (duality) between them in two dimensions. Consider a bosonic Hubbard model on a square lattice — hopping particles with point-like repulsion

\[ \hat{H} = -J \sum_{\langle i,j \rangle} (\hat{b}_i^+ \hat{b}_j + h.c) \]

\[ + \frac{U}{2} \sum_i \hat{n}_i (\hat{n}_i - 1) - J \sum_i \hat{n}_i \]

where \( \hat{n}_i = \hat{b}_i^+ \hat{b}_i \)

1) \( J = 0 \) lattice sites decouple \( \rightarrow \) single site problem

- \( n = 0 \) \( \mu < 0 \)
- \( n = 1 \) \( 0 < \mu < 0 \)
- \( n = 2 \) \( 0 < \mu < 20 \)

Integer occupation, filling changes in Mott insulator

2) \( U = 0 \) \( \rightarrow \) free boson lattice gas

Adding \( U \ll J \) \( \Rightarrow \) bosonic superfluid
Quantum phase diagram:

\[ \langle \psi_i \rangle \neq 0 \]

Closed to the quantum phase transition between the \( U(1) \) preserving MI phase and the \( U(1) \) SSB superfluid phase

\[ \hat{n} = n_0 + \delta \hat{n} \]

\[ n_0 = \frac{\tilde{F}}{U} \]

Approximate Hamiltonian:

\[ \hat{H} = -2J n_0 \sum_{\langle i,j \rangle} \cos(\phi_i - \phi_j) + \frac{U}{2} \sum (\delta \hat{n})^2 \]

Minimize the cosine term

Phase is sharp in GS

Particle number per site is uncertain

\[ \cos(\phi_i - \phi_j) \rightarrow \frac{1}{2} (\phi_i - \phi_j)^2 \]
Close to the MI/SSF quantum critical point one can write a continuum field theory of low-energy physics in two very different ways:

1) In terms of bosons $b$:

$$\mathcal{L}_{xc} = \left| \partial \mu \phi \right|^2 + m^2 \left| \phi \right|^2 + \kappa \left| \phi \right|^4$$

$m^2 < 0$ SSB superfluid phase

$m^2 > 0$ normal Mott insulator phase

as MI $\rightarrow$ SF, global $U(1)$ symmetry undergoes SSB at $T = 0$

2) In terms of vortices $\psi$:

Start in the SSB superfluid phase

vortices are high-energy excitations in this phase, try to "condense" them

Remember however in 2d

*) two point-like vortices interact

$$V(\mathbf{r}) \sim -q_1 \cdot q_2 \ln r$$
*) Experience the Magnus force:

\[ L_x = -\frac{i}{2} \hbar s \varepsilon_{ij} X_i \dot{X}_j \]

Effective magnetic field

In 2d vortices behave like point-like charged particles that live in magnetic field \( B_\nu \sim \hbar s \), we cannot SSB condense them, but can use Higgs theory:

\[ \mathcal{L}_{\text{Ah}} = \left( \partial_\mu - i A_\mu \right) \mathbf{A}^2 + \bar{\psi} \left( i \gamma^\mu \partial_\mu - m^2 \right) \psi + \frac{1}{4} F_{\mu \nu}^2 \]

**Mott insulator**

\[ \langle \mathbf{A} \rangle = 0 \]

\[ m^2 < 0 \]

**Higgs phase**

\[ \langle \mathbf{A} \rangle = 0 \]

\[ m^2 > 0 \]

Vortices are gapped, can be ignored

2+1 dim Maxwell EO

with massless photon excitation

Goldstone of \( U(1)_s \) SSB
Dictionary of boson/wortex duality:

**XY model**
\[ j^x = b \times \delta_x \ b \]

boson \( b \)

2\pi vortex of \( b \)

\[ \phi \rightarrow -\phi \]
\[ b = \Gamma e^{i\phi} \]

**Abelian Higgs model**
\[ j^n = \frac{1}{2\pi} \epsilon_{n \rho \phi} \partial_\phi A^\rho \]

instanton of \( A^\rho \) (destroys flux of \( A \))

Numerous checks of this duality close to the critical point (Monte-Carlo, ...)

Rigorous duality was defined on a lattice

Peskin 1978

Dasgupta/Halperin 1981