Problem set 4: Screening of the Coulomb potential in metals

Due date: Dec 11

Electrons in a metal screen electrostatic potential created by external impurities. In this problem set you are asked to determine the static limit $\omega = 0$ of the screened Coulomb potential in the low-momentum regime $k \ll p_F$, where $p_F$ is the Fermi momentum. One can obtain the screened potential by summing a series of Feynman diagrams

Here the wiggly line denotes $V_k = 4\pi e^2 / k^2$ which is the Fourier transform of the bare Coulomb potential and the fermion loop $\Pi(\omega, k)$ is the polarization operator that you will be asked to compute.

1. From the figure, the screened Coulomb potential is

$$\tilde{V}_{\omega, k} = V_k + V_k \Pi(\omega, k) V_k + V_k \Pi(\omega, k) V_k \Pi(\omega, k) V_k + \ldots$$

2. To determined the static limit $\omega = 0$ of the screened potential we must calculate the static polarization operator $\Pi(\omega = 0, k)$. It is given by the loop integral

$$\Pi(\omega = 0, k) = -2i \int G(\epsilon, p_-) G(\epsilon, p_+) \frac{d\epsilon d^3p}{(2\pi)^4},$$

where $p_\pm = p \pm k/2$ and we introduced the fermionic Green’s function in momentum space

$$G(\epsilon, p) = \frac{1}{\epsilon - \xi_p + i\delta \text{sign}(\xi_p)}.$$ 

3. Show that in the regime $k \ll p_F$ one has approximately

$$n(p_-) - n(p_+) = k \cos \theta \delta(p - p_F),$$

where $\theta$ is the angle between the vectors $p$ and $k$. Using this in Eq. (4) demonstrate that approximately

$$\Pi(\omega = 0, k) = -\nu,$$

where $\nu = m p_F^2 / \pi^2$ is the total density of states at the Fermi level.
4. Using now the result (6) show that the screened static Coulomb potential is

$$\tilde{V}_{\omega = 0,k} = \frac{4\pi e^2}{\kappa^2 + \kappa_\pi}$$

where $\kappa^2 = 4\pi e^2 \nu$. Fourier transform this expression to real space, how does the screened potential decay?

5. Screening also takes place in a classical plasma. Consider a static charge $\rho_0(r)$ embedded into the plasma. The resulting electrostatic potential $\Phi(r)$ satisfies the Poisson equation

$$\nabla^2 \Phi(r) = -4\pi(\rho_0(r) + \delta \rho(r)),$$

where the density of the screening charge $\delta \rho(r)$ is fixed by the electrostatic potential $\Phi(r)$ via the Boltzmann equation

$$\delta \rho(r) = e^n(e^{-e\Phi(r)/T} - 1).$$

Linearize the resulting Poisson equation and determine the screened potential of a point charge $\rho_0(r) \sim \delta(r)$. Compare the result with the one obtained above for a zero-temperature metal.